

SOME ESTIMATES FOR A GENERALIZED ABREU'S EQUATION

AN-MIN LI, ZHAO LIAN, AND LI SHENG

ABSTRACT. We study a generalized Abreu equation and derive some estimates.

Keywords. generalized Abreu Equation

1. INTRODUCTION

This is one of a sequence of papers, aiming at generalizing the results of Chen, Li and Sheng to homogeneous toric bundles. In this paper we establish some estimates.

Let \mathfrak{t} be a linear space of dimension n , \mathfrak{t}^* be its dual space. We identify \mathfrak{t} and \mathbb{R}^n with coordinates $x = (x^1, \dots, x^n)$, and identify \mathfrak{t}^* and \mathbb{R}^n with coordinates $\xi = (\xi_1, \dots, \xi_n)$. Let $\Omega^* \subset \mathfrak{t}^*$ be a bounded convex domain, $u(\xi_1, \dots, \xi_n)$ be a C^∞ strictly convex function defined in Ω^* satisfying the following nonlinear fourth-order partial differential equation

$$(1.1) \quad \frac{1}{\mathbb{D}} \sum_{i,j=1}^n \frac{\partial^2 \mathbb{D} u^{ij}}{\partial \xi_i \partial \xi_j} = -A.$$

Here, $\mathbb{D}(\xi) > 0$ and $A(\xi)$ are two given smooth functions defined on $\overline{\Omega}^*$ and (u^{ij}) is the inverse of the Hessian matrix (u_{ij}) .

The equation (1.1) was introduced by Donaldson [16] and Raza [24] in the study of the scalar curvature of toric fibration, see also [22]. We call (1.1) a generalized Abreu equation. In this paper we derive some estimates for u , which will be used in our next works in the study of prescribed scalar curvature problems on homogeneous toric bundles.

2. PRELIMINARIES

Let $f = f(x)$ be a smooth and strictly convex function defined in a convex domain $\Omega \subset \mathfrak{t}$. As f is strictly convex, G_f defined by

$$G_f = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i dx^j = \sum_{i,j} f_{ij} dx^i dx^j$$

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is a Riemannian metric in Ω . The gradient of f defines a (normal) map ∇^f from \mathfrak{t} to \mathfrak{t}^* :

$$\xi = (\xi_1, \dots, \xi_n) = \nabla^f(x) = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n} \right).$$

The function u on \mathfrak{t}^*

$$u(\xi) = x \cdot \xi - f(x)$$

is called the *Legendre transform* of f . We write

$$u = L(f), \quad \Omega^* = \nabla^f(\Omega) \subset \mathfrak{t}^*.$$

Conversely, $f = L(u)$. It is well-known that $u(\xi)$ is a smooth and strictly convex function. Corresponding to u , we have the metric

$$G_u = \sum_{i,j} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} d\xi_i d\xi_j = \sum_{i,j} u_{ij} d\xi_i d\xi_j.$$

Under the normal map ∇^f , we have

$$\frac{\partial \xi_i}{\partial x^k} = \frac{\partial^2 f}{\partial x^i \partial x^k}, \quad \left(\frac{\partial^2 f}{\partial x^i \partial x^k} \right) = \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_k} \right)^{-1}, \quad \det \left(\frac{\partial^2 f}{\partial x^i \partial x^k} \right) = \det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_k} \right)^{-1}.$$

Then,

$$(\nabla^f)^*(G_u) = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i dx^j = G_f,$$

i.e., $\nabla^f : (\Omega, G_f) \rightarrow (\Omega^*, G_u)$ is locally isometric.

Let $\rho = [\det(f_{ij})]^{-\frac{1}{n+2}}$, we introduce the following affine invariants:

$$(2.1) \quad \Phi = \frac{\|\nabla \rho\|_{G_f}^2}{\rho^2} = \frac{1}{(n+2)^2} \|\nabla \log \det(u_{ij})\|_{G_u}$$

$$(2.2) \quad 4n(n-1)J = \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn} = \sum u^{il} u^{jm} u^{kn} u_{ijk} u_{lmn},$$

$$(2.3) \quad \Theta = J + \Phi.$$

In particular, let $\Omega^* = \Delta \subset \mathfrak{t}^*$ be a Delzant polytope, M be the associate toric variety. Set

$$w^i = x^i + \sqrt{-1}y^i, \quad z^i = e^{w^i/2}.$$

Let ω_f be a Kähler metric with a local potential function f . The Ricci curvature and the scalar curvature of ω_f are given by

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} (\log \det(f_{k\bar{l}})), \quad \mathcal{S} = \sum f^{i\bar{j}} R_{i\bar{j}},$$

respectively. When we use the log-affine coordinates, the Ricci curvature and the scalar curvature can be written as

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial x^i \partial x^j} (\log \det (f_{kl})), \quad \mathcal{S} = -\sum f^{ij} \frac{\partial^2}{\partial x^i \partial x^j} (\log \det (f_{kl})).$$

Define

$$(2.4) \quad \mathcal{K} = \|\text{Ric}\|_f + \|\nabla \text{Ric}\|_f^{\frac{2}{3}} + \|\nabla^2 \text{Ric}\|_f^{\frac{1}{2}}.$$

Put

$$R_{ij} = -\frac{\partial^2}{\partial x^i \partial x^j} (\log \det (f_{kl})), \quad \mathcal{S} = -\sum f^{ij} \frac{\partial^2}{\partial x^i \partial x^j} (\log \det (f_{ij})).$$

In terms of ξ and $u(\xi)$ the scalar curvature can be written as

$$(2.5) \quad \sum_{i,j=1}^n \frac{\partial^2 u^{ij}}{\partial \xi_i \partial \xi_j} = -\mathcal{S}.$$

(2.5) is called the Abreu's Equation.

For a given smooth function $\mathbb{D}(\xi) > 0$ defined on $\overline{\Omega}^*$ we define an operator $\mathcal{S}_{\mathbb{D}}$ as

$$(2.6) \quad \mathcal{S}_{\mathbb{D}}(u) = -\frac{1}{\mathbb{D}} \sum_{i,j=1}^n \frac{\partial^2 \mathbb{D} u^{ij}}{\partial \xi_i \partial \xi_j}.$$

Let $A(\xi)$ be given smooth function defined on $\overline{\Omega}^*$, we study the generalized Abreu equation.

$$(2.7) \quad \frac{1}{\mathbb{D}} \sum_{i,j=1}^n \frac{\partial^2 \mathbb{D} u^{ij}}{\partial \xi_i \partial \xi_j} = -A.$$

Set

$$\mathbb{F} := \frac{\mathbb{D}}{\det(u_{ij})}, \quad U^{ij} = \det(u_{kl}) u^{ij}.$$

Through the (normal) map ∇u we can view \mathbb{D} and \mathbb{F} to be smooth functions on the toric variety M .

Lemma 2.1. *The generalized Abreu equation (2.7) is equivalent to any of the following equations:*

$$(2.8) \quad \sum_{i,j} U^{ij} \mathbb{F}_{ij} = -\mathbb{D}A,$$

$$(2.9) \quad \sum_{i,j} u^{ij} \frac{\partial^2 (\log \mathbb{F})}{\partial \xi_i \partial \xi_j} + \sum_{i,j} u^{ij} \frac{\partial (\log \mathbb{F})}{\partial \xi_i} \frac{\partial (\log \mathbb{F})}{\partial \xi_j} = -A,$$

$$(2.10) \quad \mathcal{S}_{\mathbb{D}}(f) := - \sum_{i,j} f^{ij} \frac{\partial^2(\log \mathbb{F})}{\partial x^i \partial x^j} - \sum_{i,j} f^{ij} \frac{\partial(\log \mathbb{F})}{\partial x^i} \frac{\partial(\log \mathbb{D})}{\partial x^j} = A.$$

Proof. Since $\sum_i U_i^{ij} = 0$, the equation (2.7) can be written as (2.8) and (2.9). Note that

$$\begin{aligned} \sum u^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} &= \sum f^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - \sum f^{ij} \frac{\partial}{\partial x^i} \log \det(f_{kl}) \frac{\partial}{\partial x^j} \\ &= \sum f^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - \sum f^{ij} \frac{\partial \log \mathbb{F}}{\partial x^i} \frac{\partial}{\partial x^j} + \sum f^{ij} \frac{\partial \log \mathbb{D}}{\partial x^i} \frac{\partial}{\partial x^j}. \end{aligned}$$

We can re-write (2.9) in the coordinates (x^1, \dots, x^n) as (2.10). \square

Let ω_g be the Guillemin metric with local potential function \mathbf{g} . For any T^n -invariant metric $\omega \in [\omega_g]$ with local potential function \mathbf{f} , there is a function ϕ globally defined on M such that

$$\mathbf{f} = \mathbf{g} + \phi.$$

Set

$$\mathcal{C}^\infty(M, \omega_g) = \{\mathbf{f} | \mathbf{f} = \mathbf{g} + \phi, \phi \in C_{\mathbb{T}^2}^\infty(M) \text{ and } \omega_{\mathbf{g}+\phi} > 0\}.$$

Fix a large constant $K_o > 0$. We set

$$\mathcal{C}^\infty(M, \omega_g; K_o) = \{\mathbf{f} \in \mathcal{C}^\infty(M, \omega_g) | |\mathcal{S}_{\mathbb{D}}(f)| \leq K_o\}.$$

Choose a local coordinate system z^1, \dots, z^n . Denote

$$H := \frac{\det(g_{i\bar{j}})}{\det(f_{i\bar{j}})}.$$

It is known that H is a global function defined on M . Set

$$\mathbb{H} := \frac{\mathbb{F}_g}{\mathbb{F}_f} = \frac{\mathbb{D}_g}{\mathbb{D}_f} H,$$

where

$$\mathbb{D}_g := (\nabla^g)^* \mathbb{D}, \quad \mathbb{D}_f := (\nabla^f)^* \mathbb{D}, \quad \mathbb{F}_g = \mathbb{D}_g \det(g_{ij}), \quad \mathbb{F}_f = \mathbb{D}_f \det(f_{ij}).$$

3. UPPER BOUND OF H

In this section we assume that $\Omega^* = \Delta \subset t^*$ is a Delzant polytope, and M is the associate toric variety. Let $\mu : M \rightarrow \bar{\Delta} \subset \mathfrak{t}^*$ be the moment map. We introduce notations:

$$\begin{aligned} R_g &= \max_M \left\{ \left| \sum g^{ij} (\log \mathbb{F}_g)_{ij} \right| + |\nabla_x \log \mathbb{F}_g|^2 \right\}, \\ \mathcal{D} &= \max_{\bar{\Delta}} \{ |\nabla_\xi \log \mathbb{D}|, \} \quad \mathcal{R} := \max \{ R_g, \mathcal{D}^2, (\text{diam}(\Delta))^2 \}, \end{aligned}$$

where $|\nabla_x \log \mathbb{F}_g|^2 = \sum \left(\frac{\partial \log \mathbb{F}_g}{\partial x^i} \right)^2$ and $|\nabla_\xi \log \mathbb{D}|^2 = \sum \left(\frac{\partial \log \mathbb{D}}{\partial \xi_i} \right)^2$. We prove

Theorem 3.1. *For any $\phi \in C^\infty(M, \omega_g)$ we have*

$$(3.1) \quad H \leq C \exp \left\{ (2\mathcal{R} + 1) (\max_M \{\phi\} - \min_M \{\phi\}) \right\},$$

where C is a constant depending only on n , $\max |\mathcal{S}_{\mathbb{D}}(f)|$ and \mathcal{R} . Here we denote $\mathcal{S}_{\mathbb{D}}(f) = \mathcal{S}_{\mathbb{D}}(\phi)$.

Proof. Consider the function

$$\mathcal{F} := \exp\{-C\phi\}\mathbb{H},$$

where C is a constant to be determined later. \mathcal{F} attains its maximum at a point $p^* \in M$. We have, at p^* ,

$$(3.2) \quad -C \frac{\partial \phi}{\partial z^i} + \frac{\partial \log \mathbb{H}}{\partial z^i} = 0,$$

$$(3.3) \quad -C f^{i\bar{j}} \phi_{i\bar{j}} + f^{i\bar{j}} (\log \mathbb{H})_{i\bar{j}} \leq 0.$$

If $\mu(p^*) \in \Delta$ we use the affine log coordinates. Note that for any smooth function F depending only on x we have

$$\frac{1}{2} \frac{\partial F}{\partial x^i} = \frac{\partial F}{\partial w^i} = \frac{\partial F}{\partial z^i} \frac{\partial z^i}{\partial w^i} = \frac{z^i}{2} \frac{\partial F}{\partial z^i},$$

If $\mu(p^*) \in \partial\Delta$, we can choose q^* such that $\mu(q^*) \in \Delta$ very close to p^* such that, at q^* ,

$$(3.4) \quad |-C\phi_i + (\log \mathbb{H})_i| \leq \epsilon,$$

$$(3.5) \quad -C f^{ij} \phi_{ij} + f^{ij} (\log \mathbb{H})_{ij} \leq \epsilon.$$

where $F_i = \frac{\partial F}{\partial x^i}$, $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$, etc. From (3.4) we get

$$(3.6) \quad |(\log \mathbb{F}_f)_i - (\log \mathbb{F}_g)_i + C(f - g)_i| \leq \epsilon.$$

Inserting (2.10) and (3.6) into (3.5) we have

$$(3.7) \quad \begin{aligned} & C f^{ij} g_{ij} - Cn + A + \sum_{i,j} f^{ij} (\log \mathbb{D}_f)_i (\log \mathbb{F}_g)_j + \sum f^{ij} (\log \mathbb{F}_g)_{ij} \\ & - C \sum_{i,j} |f^{ij} (\log \mathbb{D}_f)_i (f - g)_j| \leq \epsilon (1 + \sum_{i,j} |f^{ij} (\log \mathbb{D}_f)_i|). \end{aligned}$$

By an orthogonal transformation we can choose the coordinates x^1, \dots, x^n such that, at q^* ,

$$f_{ij} = \lambda_i \delta_{ij}, \quad g_{ij} = \mu_i \delta_{ij}.$$

Note that

$$\begin{aligned} \left| \sum f^{ij} (\log \mathbb{D}_f)_i \right| &= \left| \frac{\partial}{\partial \xi_j} (\log \mathbb{D}) \right| \leq \mathcal{D}, \\ \left| 2 \sum_{i,j} f^{ij} (\log \mathbb{D}_f)_i (\log \mathbb{F}_g)_j \right| &= \left| 2 \sum_{i,j} \frac{\partial \log \mathbb{D}}{\partial \xi_j} (\log \mathbb{F}_g)_j \right| \leq 2\mathcal{R} \\ \left| \frac{\partial}{\partial x^j} (f - g) \right| &\leq \text{diam}(\Delta), \quad \left| \sum g^{ij} (\log \mathbb{F}_g)_{ij} \right| \leq \mathcal{R}. \end{aligned}$$

From (3.7), we get

$$C \left(\frac{\mu_1}{\lambda_1} + \cdots + \frac{\mu_n}{\lambda_n} \right) - Cn + A - \left(\frac{\mu_1}{\lambda_1} + \cdots + \frac{\mu_n}{\lambda_n} \right) \mathcal{R} - C_1 \leq \epsilon(1 + n\mathcal{D})$$

for some constant $C_1 = 3\mathcal{R}$. We choose $C = 2\mathcal{R} + 1$ and apply an elementary inequality

$$\frac{1}{n} \left(\frac{\mu_1}{\lambda_1} + \cdots + \frac{\mu_n}{\lambda_n} \right) \geq \left(\frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} \right)^{1/n}$$

to get

$$n(\mathcal{R} + 1)H^{\frac{1}{n}} \leq n(2\mathcal{R} + 1) + |A| + C_1,$$

where we use the arbitrariness of ϵ . As \mathbb{D} is bounded below and above on $\bar{\Delta}$, it follows that, at p^* ,

$$\exp\{-C\phi\}\mathbb{H} \leq C \left(2 + \frac{|A| + 3\mathcal{R}}{n(\mathcal{R} + 1)} \right)^n \exp\{-(2\mathcal{R} + 1) \min_M \{\phi\}\}.$$

for some constant $C > 0$. Then (3.2) follows. \square

Donaldson [15] derived a L^∞ estimate for the Abreu's equation and $n = 2$. His method can be applied directly to the generalized Abreu Equation (see also [22]).

Theorem 3.2. *Let $n = 2$ and $\Delta \subset \mathfrak{t}^*$ be a Delzant polytope, $\mathbb{D} > 0$ and A be two smooth functions defined on $\bar{\Delta}$. Let $u \in C^\infty(\Delta, v)$ satisfying (1.1). Suppose that Δ is (\mathbb{D}, A, λ) -stable. Then there is a constant $C_o > 0$, depending on $\lambda, \Delta, \mathbb{D}$ and $\|\mathcal{S}_{\mathbb{D}}(u)\|_{C^0}$, such that $|\max_{\bar{\Delta}} u - \min_{\bar{\Delta}} u| \leq C_o$.*

Combining Theorem 3.1 and Theorem 3.2 we get

Theorem 3.3. *Let $n = 2$ and $\Delta \subset \mathfrak{t}^*$ be a Delzant polytope, M be the associate toric variety. Given two smooth functions $\mathbb{D} > 0$ and A defined on $\bar{\Delta}$. Let $u \in \mathbf{S}$ be a solution of the generalized Abreu's Equation (1.1). Suppose that Δ is (\mathbb{D}, A, λ) stable. Then there is a constant $C_1 > 0$ depending on $\text{diam}(\Delta), \mathbb{D}$ and λ , such that the following holds everywhere on M*

$$(3.8) \quad H \leq C_1.$$

4. ESTIMATES OF THE DETERMINANT

Definition 4.1. A convex domain $\Omega \subset \mathbb{R}^n$ is called normalized when its center of mass is 0 and $n^{-\frac{3}{2}}D_1(0) \subset \Omega \subset D_1(0)$.

Lemma 4.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain, and f be a smooth and strictly convex function defined on Ω with

$$f|_{\partial\Omega} = C, \quad \inf_{\Omega} f = 0.$$

Assume that f satisfies the equation (2.10). Suppose that

$$(4.1) \quad \sup_{\Omega} |\nabla f| \leq N_1, \quad \sup_{\Omega} |\nabla_{\xi} \log \mathbb{D}| \leq N_1$$

for some constant $N_1 > 0$. Then the following estimate holds:

$$(4.2) \quad \exp \left\{ -\frac{4C}{C-f} \right\} \det(f_{ij}) \leq C_2, \quad \forall x \in \Omega,$$

where C_2 is a constant depending only on C , N_1 and $\max_{\Omega} A$.

Proof. Consider the function

$$(4.3) \quad F := \exp \left\{ -\frac{4C}{C-f} + \epsilon \sum \left(\frac{\partial f}{\partial x^i} \right)^2 \right\} \mathbb{F}.$$

Clearly, F attains its supremum at some interior point p^* . At p^* , we have,

$$(4.4) \quad \begin{aligned} & -gf_i + 2\epsilon \sum f_j f_{ij} + (\log \mathbb{F})_i = 0 \\ & -2g - g' \sum f^{ij} f_i f_j + 2\epsilon \sum f_{ii} + 2\epsilon \sum f_i f^{kl} f_{kli} \\ & + \sum f^{ij} (\log \mathbb{F})_{ij} \leq 0. \end{aligned}$$

where $g = \frac{4C}{(C-f)^2}$. By (4.4), at p^* , we have

$$\begin{aligned} 2\epsilon \sum f_i f^{kl} f_{kli} &= 2\epsilon \sum f_i \frac{\partial \log \det(f_{kl})}{\partial x^i} = 2\epsilon \sum f_i \frac{\partial \log \mathbb{F}}{\partial x^i} - 2\epsilon \sum f_i f_{ij} \frac{\partial \log \mathbb{D}}{\partial \xi_j} \\ &= 2\epsilon g \sum f_i^2 - 4\epsilon^2 \sum f_{ij} f_i f_j - 2\epsilon \sum f_i f_{ij} \frac{\partial \log \mathbb{D}}{\partial \xi_j}, \end{aligned}$$

and

$$\begin{aligned} \sum f^{ij} (\log \mathbb{F})_{ij} &= -A - \sum f^{ij} (\log \mathbb{F})_i (\log \mathbb{D})_j = -A - \sum (\log \mathbb{F})_i \frac{\partial \log \mathbb{D}}{\partial \xi_i} \\ &= -A + 2\epsilon f_i f_{ij} \frac{\partial \log \mathbb{D}}{\partial \xi_j} - \sum g f_i \frac{\partial \log \mathbb{D}}{\partial \xi_i}. \end{aligned}$$

Inserting these into (4.5), using (4.1), choosing $\epsilon < \frac{1}{4N_1^2}$, we have

$$-2g - N_1^2 g' \sum f^{ii} + \epsilon \sum f_{ii} - A - N_1^2 g \leq 0.$$

Let λ_1, λ_2 be the eigenvalues of (f_{ij}) . Then we have

$$(4.6) \quad -2g - N_1^2 g' \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} + \epsilon(\lambda_1 + \lambda_2) - A - N_1^2 g \leq 0.$$

Since $\frac{\lambda_1 + \lambda_2}{2} \geq \sqrt{\lambda_1 \lambda_2}$, one can easily obtain that

$$(4.7) \quad \epsilon \lambda_1 \lambda_2 \leq (2g + \max_{\Omega} A + N_1^2 g) \sqrt{\lambda_1 \lambda_2} + 2N_1^2 g'.$$

By $\exp\{-\frac{4C}{C-f}\}(g^2 + g') \leq C_2$ and Cauchy inequalities we have

$$F \leq C_3$$

where $C_3 > 0$ is a constant depending only on N_1 and $\max_{\Omega} A$. Then the lemma follows. \square

By the same method in [7] we can prove the following two lemmas. For reader's convenience we sketch the proofs here.

Lemma 4.3. *Let $\Omega^* \subset \mathbb{R}^n$ be a normalized convex domain, and $\mathbb{D} > 0$, A be two smooth functions defined on $\bar{\Omega}^*$. Let u be a smooth and strictly convex function defined in Ω^* with*

$$u|_{\partial\Omega^*} = C, \quad \inf_{\Omega^*} u = u(p) = 0.$$

Let f be the Legendre transformation of u . Assume that u satisfies the generalized Abreu equation equation (2.7). Then there is a constant $d \geq 1$, independent of u , such that

$$(4.8) \quad \exp\left\{-\frac{4C}{C-u}\right\} \frac{\det(u_{ij})}{(d+f)^{2n}} \leq C_3$$

for some constant $C_3 > 0$ depending only on $n, \max|A|$ and C .

Proof. We can show as in [7] that there are constants $d \geq 1, b > 0$ such that

$$\frac{\sum (x^k)^2}{(d+f)^2} \leq b.$$

Consider the following function

$$F = \exp\left\{-\frac{m}{C-u} + L\right\} \frac{1}{\mathbb{F}(d+f)^{2n}},$$

where

$$L = \epsilon \frac{\sum (x^k)^2}{(d+f)^2}.$$

m and ϵ are positive constants to be determined later. Clearly, F attains its supremum at some interior point p^* of Ω^* . We have, at p^* ,

$$(4.9) \quad F_i = 0,$$

$$(4.10) \quad \sum u^{ij} F_{ij} \leq 0,$$

where we denote $F_i = \frac{\partial F}{\partial \xi_i}$, $F_{ij} = \frac{\partial^2 F}{\partial \xi_i \partial \xi_j}$, $f_i = \frac{\partial f}{\partial \xi_i}$ and so on. Using the PDE (2.8) we calculate both expressions (4.9) and (4.10) explicitly:

$$(4.11) \quad -\frac{m}{(C-u)^2} u_i + L_i - 2n \frac{f_i}{d+f} - (\log \mathbb{F})_i = 0,$$

and

$$(4.12) \quad -\frac{2m}{(C-u)^3} \sum u^{ij} u_i u_j - \frac{mn}{(C-u)^2} + \sum u^{ij} L_{ij} - 2n \frac{\sum u^{ij} f_{ij}}{d+f} + 2n \frac{\sum u^{ij} f_i f_j}{(d+f)^2} + \frac{\sum u^{ij} \mathbb{F}_i \mathbb{F}_j}{\mathbb{F}^2} + A \leq 0.$$

We choose $\epsilon = \frac{1}{8000n^2b}$, $m = 4C$. Note that $\frac{1}{C_1} \leq \mathbb{D} \leq C_1$ for some constant $C_1 > 0$. By the same calculation in [7] we get

$$\exp \left\{ -\frac{m}{C-u} + \epsilon \frac{\sum (x^k)^2}{(d+f)^2} \right\} \frac{1}{(d+f)^{2n} \mathbb{F}} \leq d_1$$

for some constant $d_1 > 0$ depending on n, b and \mathbb{D} . Since F attains its maximum at p^* , the Lemma follows. \square

Let $\Delta \subset \mathbb{R}^2$ be the Delant polytope defined by

$$\Delta = \{ \xi | l_i(\xi) > 0, \quad 0 \leq i \leq d-1 \}$$

where $l_i(\xi) := \langle \xi, \nu_i \rangle - \lambda_i$ and ν_i is the inward pointing normal vector to the edge ℓ_i of Δ . Set $v = \sum_i l_i \log l_i$ and

$$C^\infty(\Delta, v) = \{ u | u = v + \psi \text{ is strictly convex, } \psi \in C^\infty(\bar{\Delta}) \}.$$

Lemma 4.4. *Let $\Delta \subset \mathbb{R}^2$ be a Delzant ploytope, $\mathbb{D} > 0$ and A be smooth functions on $\bar{\Delta}$. Suppose that $0 \in \Delta^\circ$, and $u \in C^\infty(\Delta, v)$ satisfies the generalized Abreu equation (2.7) and*

$$u(0) = \inf u, \quad \nabla u(0) = 0.$$

Then,

$$\frac{\det(u_{ij})}{(d+f)^4}(p) \leq C_4 (d_E(p, \partial\Delta))^4,$$

where C_4 is a positive constant depending $\text{diam}(\Delta)$, $\max_{\bar{\Delta}} \mathbb{D}$, $\min_{\bar{\Delta}} \mathbb{D}$ and $\|A\|_{L^\infty(\Delta)}$. Here $d_E(p, \partial\Delta)$ denotes the Euclidean distance from p to $\partial\Delta$.

Proof. We can find constants $d \geq 1, b > 0$ such that

$$\frac{\sum (x^k)^2}{(d+f)^2} \leq b.$$

Consider the following function

$$F = (r^2 - \gamma)^2 \exp \{L\} \frac{1}{\mathbb{F}(d+f)^4},$$

in $D_r(q)$, where $\overline{D_r(q)} \subset \bar{\Delta}$, and

$$\gamma = \sum (\xi_i - \xi_i(q))^2, \quad L = \epsilon \frac{\sum (x^k)^2}{(d+f)^2},$$

and ϵ is a positive constants to be determined later. Since $d_E(p, \partial D_r(q)) \leq d_E(p, \partial \Delta)$, by Proposition 2 in [13], we have

$$\lim_{p \rightarrow \partial D_r(q)} \det(u_{ij}) d_E^2(p, \partial D_r(q)) = 0.$$

Then, F attains its supremum at some interior point p^* of $D_r(q)$. At p^* , we have,

$$(4.13) \quad L_i - \frac{4f_i}{d+f} - \frac{2\gamma_i}{r^2 - \gamma} - (\log \mathbb{F})_i = 0,$$

and

$$(4.14) \quad \begin{aligned} & \sum u^{ij} L_{ij} - 4 \frac{\sum u^{ij} f_{ij}}{d+f} + 4 \frac{\sum u^{ij} f_i f_j}{(d+f)^2} \\ & - \frac{2 \sum u^{ij} \gamma_{ij}}{r^2 - \gamma} - \frac{2 \sum u^{ij} \gamma_i \gamma_j}{(r^2 - \gamma)^2} + \frac{\sum u^{ij} \mathbb{F}_i \mathbb{F}_j}{\mathbb{F}^2} + A \leq 0. \end{aligned}$$

We choose $\epsilon = \frac{1}{2000b}$. By the same calculation in [7] we prove the lemma. \square

5. INTERIOR ESTIMATE OF Θ

Let $\Delta \subset \mathbb{R}^2$ be the Delant polytope. The purpose of this section is to estimate Θ near the boundary $\partial \Delta$. We use the blow-up analysis to prove the following result.

Theorem 5.1. *Let $u \in C^\infty(\Delta, v)$. Choose a coordinate system (ξ_1, ξ_2) such that $\ell = \{\xi | \xi_1 = 0\}$. Denote $B_b(p, \Delta) = \{q \in \Delta | d_u(q, p) < b\}$, where $d_u(p, q)$ is the distance from p to q with respect to the Calabi metric G_u . Let $p \in \ell^\circ$ such that $B_b(p, \Delta)$ intersection with $\partial \Delta$ lies in the interior of ℓ . Suppose that*

$$(5.1) \quad \|\mathcal{S}_{\mathbb{D}}(u)\|_{C^3(B_b(p, \Delta))} \leq N_2, \quad h_{22}|_\ell \geq N_2^{-1},$$

for some constant $N_2 > 0$, where $h = u|_\ell$ and $\|\cdot\|_{C^3(\Delta)}$ denotes the Euclidean C^3 -norm. Then, for any $p \in B_{b/2}(p, \Delta)$,

$$(5.2) \quad (\Theta + \mathcal{K})(p) d_u^2(p, \ell) \leq C_5,$$

where C_5 is a positive constant depending only on N_2 .

To prove this theorem we need the following theorems, the proofs can be found in [20] and [9].

Theorem 5.2. [Li-Jia] Let $u(\xi_1, \xi_2)$ be a C^∞ strictly convex function defined in a convex domain $\Omega \subset \mathbb{R}^2$. If

$$\mathcal{S}(u) = 0, \quad u|_{\partial\Omega} = +\infty,$$

then the graph of u must be an elliptic paraboloid.

Remark 5.3. If we only assume that $u \in C^5$, Theorem 5.2 remains true.

Theorem 5.4. Suppose that u_k is a sequence of smooth strictly convex functions on $\Omega^* \subset \mathbb{R}^n$ containing 0 and $\Theta_{u_k} \leq N_3^2$, and that u_k are already normalized such that

$$u_k \geq u_k(0) = 0, \quad \frac{\partial^2 u_k}{\partial \xi_i \partial \xi_j}(0) = \delta_{ij}.$$

Then

- (a) there exists a constant $a > 0$ such that $B_{a, u_k}(0) \subset \Omega$,
- (b) there exists a subsequence of u_k that locally C^2 -converges to a strictly convex function u_∞ in $B_{a, u_\infty}(0)$,
- (c) moreover, if (Ω^*, G_{u_k}) is complete, (Ω^*, G_{u_∞}) is complete.

Theorem 5.5. Let $\Omega^* \subset \mathbb{R}^n$ be a convex domain. Let u be a smooth strictly convex function on Ω^* with $\Theta \leq N_3^2$. Then (Ω^*, G_u) is complete if and only if the graph of u is Euclidean complete in \mathbb{R}^{n+1} .

Proof of Theorem 5.1. If the theorem is not true, then there exists a sequence of functions u_k and a sequence of points $p_k \in B_{b/2}(p_k, \Delta)$ such that

$$(5.3) \quad \Theta_{u_k}(p_k) d_{u_k}^2(p_k, \ell) \rightarrow \infty.$$

Let $B^{(k)}$ be the $\frac{1}{2}d_{u_k}(p_k, \ell)$ -ball centered at p_k and consider the affine transformationally invariant function

$$F_k(p) = \Theta_{u_k}(p) d_{u_k}^2(p, \partial B^{(k)}).$$

F_k attains its maximum at p_k^* . Put

$$d_k = \frac{1}{2}d_{u_k}(p_k^*, \partial B^{(k)}).$$

By adding linear functions we assume that

$$u_k(p_k^*) = 0, \quad \nabla u_k(p_k^*) = 0.$$

Let l_k be the largest constant such that the section $S_{u_k}(p_k^*, l_k)$ is compact. Denote by h_k the restriction of u_k to ℓ . Then, h_k locally uniformly converges to a convex function

h on ℓ , and u_k locally C^6 -converges in Δ to a strictly convex function u_∞ . u_∞ can be continuously extended to be defined on $\bar{\Delta}$. By Lemma 3.3 in [21], C^0 -estimate and Lemma 4.4 we have

$$-C_1 - \log d_E(p, \partial\Delta) \leq \log \det(D^2 u_k)(p) \leq C_1 - 8 \log d_E(p, \partial\Delta)$$

and

$$(5.4) \quad \frac{\partial^2 h_k}{\partial \xi_2^2} \geq C_2$$

for some positive constants C_1, C_2 independent of k . As in [10] we can conclude that

$$(5.5) \quad u_\infty(q) = h(q), \quad \forall q \in \ell^\circ$$

Using this and the interior regularity we have $\lim_{k \rightarrow \infty} l_k = 0$. Then it follows from (5.4) and (5.5) that

$$(5.6) \quad \lim_{k \rightarrow \infty} \text{diam}(S_{u_k}(p_k^*, l_k)) = 0.$$

By taking a proper coordinate translation we may assume that the coordinate of p_k^* is 0. Then

- $\Theta_{u_k}(0) d_k^2 \rightarrow \infty$.
- $\Theta_{u_k} \leq 4\Theta_{u_k}(0)$ in $B_{d_k}^{(k)}(0)$.

By (5.3) we have

$$(5.7) \quad \lim_{k \rightarrow \infty} \Theta_{u_k}(0) = +\infty.$$

We take an affine transformation on u_k :

$$\xi^{*k} = A_k \xi, \quad u_k^*(\xi^{*k}) := \lambda_k u_k(A_k^{-1} \xi^{*k}),$$

where $\lambda_k = \Theta_{u_k}(0)$. Choose A_k such that $\frac{\partial^2 u_k^*}{\partial \xi_i \partial \xi_j}(0) = \delta_{ij}$. Denote $A_k^{-1} = (b_{ij}^k)$. Then by the affine transformation rule we know that $\Theta_{u_k^*}(0) = 1$, and for any fixed large R , when k large enough

$$\Theta_{u_k^*} \leq 4 \quad \text{in} \quad B_R^{(k)}(0).$$

Denote by d_k^* the change of d_k after the affine transformation. Then

$$\lim_{k \rightarrow \infty} d_k^* = +\infty.$$

By Theorem 5.4, one concludes that u_k^* locally uniformly C^2 -converges to a function u_∞^* , and the graph of u_∞^* is complete with respect to the Calabi metric $G_{u_\infty^*}$. By Theorem 5.5 the graph of u_∞^* is Euclidean complete. We have, in $B_{R, u_\infty^*}(0)$,

$$(5.8) \quad C_2^{-1} \leq \lambda_{\min}(u_k^*) \leq \lambda_{\max}(u_k^*) \leq C_2,$$

where $C_2 > 0$ is a constant depending only on R . Here $\lambda_{\min}(u_k^*)$ and $\lambda_{\max}(u_k^*)$ denotes the minimal and maximal eigenvalues of the Hessian of u_k^* in $B_{R, u_\infty^*}(0)$. Then there exists an Euclidean ball $D_\epsilon(0)$ such that $D_\epsilon(0) \subset A_k(S_{u_k}(0, l_k))$ when k large enough. Therefore $A_k^{-1}D_\epsilon(0) \subset S_{u_k}(0, l_k)$. It follows from (5.6) that for any $1 \leq i, j \leq 2$

$$(5.9) \quad \lim_{k \rightarrow \infty} \left| \frac{\partial \xi_i}{\partial \xi_j^{*k}} \right| = \lim_{k \rightarrow \infty} |b_{ij}^k| = 0,$$

where $A_k^{-1} = (b_{ij}^k)$. By a direct calculation we have

$$\left| \frac{\partial \mathcal{S}_{\mathbb{D}_k}(u_k^*)}{\partial \xi_i^{*k}} \right| = \lambda_k^{-1} \left| \sum_j b_{ji}^k \frac{\partial \mathcal{S}_{\mathbb{D}_k}(u_k)}{\partial \xi_j} \right| \leq 8 \text{diam}(\Omega) \epsilon^{-1} \lambda_k^{-1} K_o \rightarrow 0$$

as k goes to infinity. Similarly, we have

$$\lim_{k \rightarrow \infty} \|\mathcal{S}_{\mathbb{D}_k}(u_k^*)\|_{C^2} = 0, \quad \lim_{k \rightarrow \infty} \max |\nabla_{\xi^{*k}} \log \mathbb{D}_k| = 0.$$

By (5.8) there is a constant $r_1 > 0$ independent of k such that $S_{u_k^*}(q, r_1) \subset B_{R, u_\infty^*}(0)$ for any $q \in B_{R/2, u_\infty^*}(0)$. It follows from Theorem 6.1 below that u_k^* locally $C^{3, \alpha}$ -converges to a function u_∞^* . Then by the standard elliptic equation technique we obtain that u_k^* locally C^5 -converges to u_∞^* with

$$\mathbb{D} = \text{constant}, \quad \Theta_{u_\infty^*}(0) = 1.$$

Hence by Theorem 5.2 and Remark 5.3, u_∞^* must be quadratic and $\Theta \equiv 0$. We get a contradiction. ■

By the same argument of [9] we have the following corollaries.

Corollary 5.6. *Let $u \in C^\infty(\Delta, v)$. Suppose that*

$$(5.10) \quad \max_{\Delta} u - \min_{\Delta} u \leq N_2, \quad \|\mathcal{S}_{\mathbb{D}}(u)\|_{C^3(\Delta)} \leq N_2$$

for some constant $N_2 > 0$. Then, for any $p \in \Delta$,

$$(5.11) \quad (\Theta + \mathcal{K})(p) d_u^2(p, \partial \Delta) \leq C_5,$$

where C_5 is a positive constant depending only on N_2 .

Corollary 5.7. *Let u be as that in Corollary 5.6, and $\mathbb{A}(u)$ be operator such that $\mathbb{A}(u) d^2(p, \partial \Delta)$ is affine invariant. Suppose that*

$$\|\mathbb{A}(u)\|_{C^3(\Delta)} \leq N_2, \quad |\mathbb{A}(u)| \geq \delta > 0.$$

Then there is a constant $C_5 > 0$, depending only on Δ and N_2 , such that

$$(5.12) \quad \|\nabla \log |\mathbb{A}(u)|\|^2(p) d_u^2(p, \partial \Delta) \leq C_5, \quad \forall p \in \Omega.$$

6. A CONVERGENCE THEOREM AND ITS APPLICATION

6.1. A convergence theorem. In this section we extend the Theorem 3.6 in [9] to the generalized Abreu equation. Denote by $\mathcal{F}(\Omega^*, C)$ the class of smooth convex functions defined on Ω^* such that

$$\inf_{\Omega^*} u = 0, \quad u = C > 0 \text{ on } \partial\Omega^*.$$

The main result of this subsection is the following convergence theorem.

Theorem 6.1. *Let $\Omega^* \subset \mathbb{R}^2$ be a normalized convex domain. Let $u_k \in \mathcal{F}(\Omega^*, 1)$ be a sequence of functions and p_k^o be the minimal point of u_k . Let $\mathbb{D}_k > 0$ be given smooth function defined on $\bar{\Omega}^*$. Suppose that there is $N_4 > 0$ such that*

$$|\mathcal{S}_{\mathbb{D}_k}(u_k)| \leq N_4, \quad N_4^{-1} \leq \mathbb{D}_k \leq N_4$$

and

$$\sup_{\Omega^*} |\nabla_{\xi} \log \mathbb{D}_k| \leq N_4.$$

Then there exists a subsequence of functions, without loss of generality, still denoted by u_k , locally uniformly converging to a function u_{∞} in Ω^ and p_k^o converging to p_{∞}^o such that*

$$d_E(p_{\infty}^o, \partial\Omega^*) > s$$

for some constant $s > 0$ and in $D_s(p_{\infty}^o)$

$$\|u\|_{C^{3,\alpha}} \leq C_6$$

for some $C_6 > 0$ and $\alpha \in (0, 1)$.

To prove Theorem 6.1, we need the following lemma, the proof can be find in [21].

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded normalized convex domain. Let $f_k \in \mathcal{F}(\Omega, C)$ be a sequence of functions satisfying the equation $\mathcal{S}_{\mathbb{D}_k}(f_k) = A_k$. Suppose that A_k (resp. \mathbb{D}_k) C^m -converges to A (resp. $\mathbb{D} > 0$) on $\bar{\Omega}$, and there are constants $0 < N_5 < N_6$ independent of k such that*

$$(6.1) \quad N_5 \leq \det \left(\frac{\partial^2 f_k}{\partial x^i \partial x^j} \right) \leq N_6$$

hold in Ω . Then there exists a subsequence of functions, without loss of generality, still denoted by f_k , locally uniformly converging to a function f_{∞} in Ω and, for any open set Ω_o with $\bar{\Omega}_o \subset \Omega$, and for any $\alpha \in (0, 1)$, f_k $C^{m+3,\alpha}$ -converges to f_{∞} in Ω_o .

Proof of Theorem 6.1 By Lemma 4.3 we have uniform estimate in $\Omega_k^{\circ} := \{\xi | u_k \leq \frac{1}{2}\}$:

$$\frac{\det(u_{ij})}{(d+f)^4} \leq e^8 C_3$$

for any one u in $\{u_k\}$. Then there is $r > 0$ independent of k such that $D_r(0) \subset \nabla u_k(\Omega_k^\circ)$, and there is a constant $d > 0$ such that

$$(6.2) \quad \det(f_{ij}) \geq \frac{1}{e^8 C_3 (r+d)^4}, \quad -1 \leq f \leq r,$$

for any one f in f_k . Then there exists a subsequence of functions, without loss of generality, still denoted by f_k , locally uniformly converging to a function f_∞ in $D_r(0)$. By (6.3) and the Alexandrov-Pogorelov theorem f_∞ is strictly convex. Then there is a constant $C_3 > 0$ such that the section

$$\bar{S}_{f_\infty}(0, C_3) := \{x | f_\infty \leq C_3\}$$

is compact. By Lemma 4.2 we have, in the section $\bar{S}_{f_\infty}(0, C_3/2)$,

$$(6.3) \quad \det(f_{ij}) \leq e^8 C_2$$

for any one f in f_k . Using Theorem 6.2 we conclude that f_k $C^{3,\alpha}$ -converges to f_∞ in the section $S_{f_\infty}(0, C_3/2)$. It follows that p_k° converges to p_∞° and there exist constants s such that $d_E(p_k^\circ, \partial\Omega) > 2s$, and u_k $C^{3,\alpha}$ -converges to u_∞ in $D_s(p_\infty^\circ)$. The theorem follows. \blacksquare

6.2. An application of Theorem 5.1. In order to use affine blow-up technique we need to control sections (see Subsection §7.2[9] and Subsection §5.3-§5.4 [10]). For the generalized Abreu equation we need more arguments due to the presence of the function \mathbb{D} . The following Theorem 6.3 will be used in our next works.

Let $\Delta \subset \mathbb{R}^2$ be a Delzant ploytope. Let ℓ be an edge of Δ and ℓ° be the interior of ℓ . Let $q \in \ell^\circ$. We fix a coordinate system on \mathfrak{t}^* such that (i) $\ell = \{\xi | \xi_1 = 0\}$, (ii) $\Delta \subset \{\xi_1 > 0\}$.

Let $u_k \in C^\infty(\Delta, v)$ be a sequence of functions with $\mathcal{S}_{\mathbb{D}}(u_k) = A_k$, let p_k° be points such that

$$(6.4) \quad d_{u_k}(p_k^\circ, \partial\Delta) = d_{u_k}(p_k^\circ, \ell^\circ) \rightarrow 0,$$

where $d_{u_k}(p_k^\circ, \partial\Delta)$ is the geodesic distance from p_k° to $\partial\Delta$ with respect to the Calabi metric G_{u_k} . Suppose that A_k C^3 -converges to A on $\bar{\Delta}$. By the interior regularity (see [21]) we have u_k locally uniformly $C^{6,\alpha}$ converges to a strictly convex function u_∞ and $\lim_{k \rightarrow \infty} d_E(p_k^\circ, \ell^\circ) = 0$. Then by the C^0 -estimate $u_k|_{\ell^\circ}$ locally uniformly to a convex function h .

Let u be one of u_k . By adding a linear function we normalize u such that p° is the minimal point of u ; i.e.,

$$(6.5) \quad u(p^\circ) = \inf u.$$

Let \check{p} be the minimal point of u on ℓ . By a coordinate translation and by adding some constant to u , we may require that

$$(6.6) \quad u(p^\circ) = 0, \quad \xi(p^\circ) = 0$$

As in the proof of Theorem 5.1 we have

$$(6.7) \quad \lim_{k \rightarrow \infty} \text{diam}(S_{u_k}(p_k^\circ, u_k(\check{p}_k) - u_k(p_k^\circ)) = 0.$$

We consider the following affine transformation on u :

$$(6.8) \quad \tilde{u}(\tilde{\xi}) = \lambda u(A^{-1}(\tilde{\xi})),$$

where $A(\tilde{\xi}_1, \tilde{\xi}_2) = \sum a_i^j \tilde{\xi}_j$. We choose λ and (a_i^j) such that, at p° ,

$$(6.9) \quad \tilde{u}_{ij}(0) = \delta_{ij}, \quad d_{\tilde{u}}(p^\circ, \partial\Delta) = 1.$$

Then we have a sequence of functions \tilde{u}_k with $\tilde{A}_k \rightarrow 0$. Denote $\mathcal{H}_k := \tilde{u}_k(\check{p}_k) - \tilde{u}_k(p_k^\circ)$.

Theorem 6.3. *Let \tilde{u}_k be a sequence of functions satisfying $\mathcal{S}_{\mathbb{D}}(u_k) = A_k$ as above. Suppose that*

$$\tilde{u}_{ij}(0) = \delta_{ij}, \quad d_{\tilde{u}}(p^\circ, \partial\Delta) = 1, \quad \tilde{u}_k(p_k^\circ) = 0, \quad \xi(p_k^\circ) = 0.$$

Then there are constants $C_8 > C_7 > 0$ independent of k such that

$$C_7 \leq \mathcal{H}_k \leq C_8.$$

Proof. We first prove $\mathcal{H}_k \geq C_7$. By Theorem 5.1 we have

$$\Theta_k \leq 4C_5, \quad \text{in } B_{\frac{1}{2}}(p_k^\circ).$$

Then by Theorem 5.4 we conclude that $u_k(\check{p}_k) - u_k(p_k^\circ) \geq C_7$.

Now we prove $\mathcal{H}_k \leq C_8$. Suppose that $u_k(\check{p}_k)$ has no upper bound. Then we can choose a sequence of constants $N_k \rightarrow \infty$ such that

$$0 < N_k < u_k(\check{p}_k), \quad \lim_{k \rightarrow \infty} N_k \max |\tilde{A}_k| = 0.$$

For each u_k we take an affine transformation $\hat{A}_k := (A_k, (N_k)^{-1})$ to get a new function \hat{u}_k , i.e.,

$$\hat{u}_k = (N_k)^{-1} \tilde{u}_k \circ (A_k)^{-1}.$$

such that $S_{\hat{u}_k}(A_k p_k^\circ, 1)$ is normalized. Then from (6.7) we conclude that

$$\lim_{k \rightarrow \infty} \sup_{S_{\hat{u}_k}(A_k p_k^\circ, 1)} |\nabla \log \mathbb{D}_k| \rightarrow 0.$$

Obviously,

$$\lim_{k \rightarrow \infty} \hat{A}_k = \lim_{k \rightarrow \infty} N_k \tilde{A}_k = 0,$$

$$\lim_{k \rightarrow \infty} d_{\hat{u}_k}(A_k p_k^\circ, A_k \ell) = \lim_{k \rightarrow \infty} \frac{1}{\sqrt{N_k}} d_{u_k}(p^\circ, \ell) = 0.$$

On the other hand, by Theorem 5.1 we conclude that \hat{u}_k locally $C^{3,\alpha}$ -converges to a strictly convex function \hat{u}_∞ in a neighborhood of the minimal point of \hat{u}_∞ . In particular, there is a constant $C_3 > 0$ such that

$$d_{\hat{u}_k}(A_k p_k^\circ, A_k \ell) \geq C_3.$$

We get a contradiction. Hence, we prove the lemma. ■

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DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, 610064, CHINA
E-mail address: anminliscu@126.com

DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, 610064, CHINA

DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, 610064, CHINA
E-mail address, Corresponding author: lshengscu@gmail.com